SCHRIFTELIJK TENTAMEN

ASTROFYSISCHE HYDRODYNAMICA

3^{rd} kwartaal 2011/2012

April 12, 2012

NOTE: THIS EXAM CONTAINS 4 QUESTIONS, 7 pages and one figure. Please assure you have read all pages and questions.
Mention your name and studentnr. on ALL pages that you hand in.

Question 1.: Shocks and Rankine-Hugoniot Conditions

Central to investigating the structure and physics of shockwaves are the Rankine-Hugoniot jump conditions. They are the four flux conservations laws that hold for an infinitely thin shock, giving us the information needed to calculate the state of the gas behind the shock (location 2), given its state just ahead of the shock (location 1).

a) With ρ the density of the gas, P the pressure of the gas, h the enthalpy of the gas, V_n the velocity component perpendicular to the shock, V_t the velocity component parallel to the shock, write down the Rankine-Hugoniot jump conditions for the shock. Use the index 1 for the state of the gas in front of the shock, index 2 for the gas state behind the shock.

b) explain the physical significance of each of the terms.

c) With $J_i = \rho_i V_{ni}$ the mass flux across the shock, and $\mathcal{V}_i = 1/\rho_i$ the specific volume show that the Rankine-Hugoniot conditions imply that

$$J^2 = \frac{(P_2 - P_1)}{(\mathcal{V}_1 - \mathcal{V}_2)} \tag{1}$$

d) In the pressure-volume (pV) diagram, indicate the socalled *Rayleigh line*, described by the equation you just derived. In the diagram, also indicate the forbidden regions and explain the rationale behind their exclusion.

e) Given that the specific enthalpy of an ideal gas is given by

$$h = \frac{\gamma P}{(\gamma - 1)\rho} = \frac{\gamma}{(\gamma - 1)} P\mathcal{V} \tag{2}$$

show that

$$\frac{1}{2}(\mathcal{V}_2 + \mathcal{V}_1)(P_2 - P_1) = \frac{\gamma}{\gamma - 1}(P_2\mathcal{V}_2 - P_1\mathcal{V}_1) \tag{3}$$

Hint: use the Rayleigh line relation.

f) ~~On the basis of the relation you just derived, infer the *Rankine-Hugoniot shock adiabat*~~

$$\frac{\gamma}{\gamma - 1}(P_2\mathcal{V}_2 - P_1\mathcal{V}_1) = \frac{1}{2}(\mathcal{V}_2 + \mathcal{V}_1)(P_2 - P_1) \tag{4}$$

Indicate the location of the shock adiabat in the (pV) diagram.

g) Explain how the combination of the Rayleigh line and Rankine-Hugoniot adiabat help to determine how the conditions in front of shock change into those behind the shock.

h) For an ideal gas, infer that the jump r in density across a shock is given by

$$r = \frac{\rho_2}{\rho_1} = \frac{\frac{\gamma+1}{\gamma-1}P_2 + P_1}{\frac{\gamma+1}{\gamma-1}P_1 + P_2} \tag{5}$$

i) What, therefore, is the maximum density jump for a very strong shock ($P_2 \gg P_1$) in a monatomic gas (with $\gamma = 5/3$) ?

j) Rewriting the Rankine-Hugoniot conditions in terms of the Mach number of the shock (strictly speaking, the Mach number of the velocity component normal to the shock),

$$\mathcal{M}_n = \frac{|V_n|}{C_s}, \tag{6}$$

(where C_s is the sound velocity) we have that

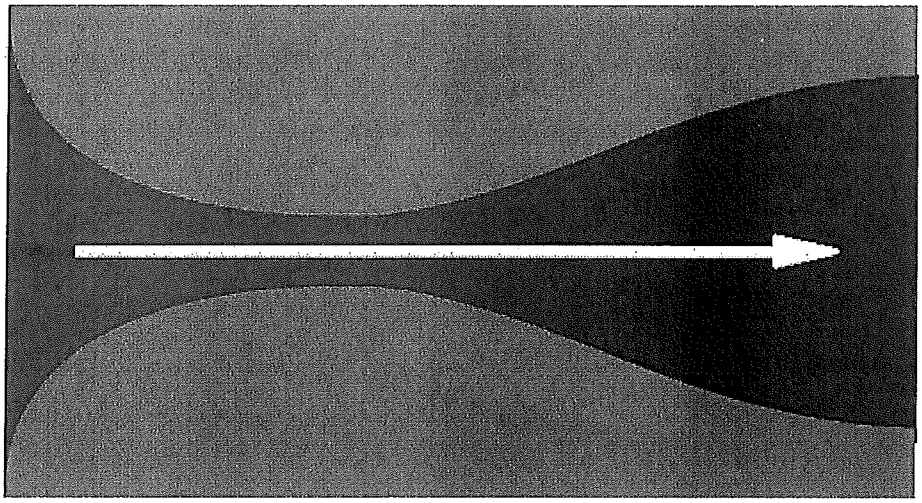
$$\frac{P_2}{P_1} = 1 + \frac{2\gamma}{\gamma+1}(M_n^2 - 1). \tag{7}$$

Infer from this relation that the pressure behind a blast wave with velocity V_s that occurs in the interstellar medium with pressure P_0 and density ρ_0, for which $M_n = M_s = V_s/c_s$

$$P_2 \approx \frac{2\gamma}{\gamma+1}M_s^2 P_0 = \frac{2}{\gamma+1}\rho_0 V_s^2. \tag{8}$$

Question 2.: Jets and the De Laval Nozzle

A very interesting application of the Bernoulli equation, for compressible fluids, concerns the *de Laval Nozzle.* A de Laval nozzle is a tube that is pinced in the middle, making a carefully balanced, asymmetric hourglass shape. The nozzle was developed in 1888 by the Swedish inventor Gustaf de Laval for use on a steam turbine. The principle was first used for rocket engines by Robert Goddard. An illustration of a de Laval Nozzle is shown in figure 1.



Figuur 1: Illustration of the de Laval Nozzle

a. We make the approximation of steady, quasi-1-D barotropic flow. Essential is that the flow is compressible (ie. not incompressible). The 1-D flow velocity (along the x-axis) is u, the density is ρ, the pressure p. Write the Bernoulli equation for compressible flow (ignoring an external force like gravity).

b. If the local sectional area of the nozzle is A, write the continuity equation.

c. Infer from the Bernoulli equation that

$$\frac{d\rho}{\rho} = -M^2 \frac{du}{u} \tag{9}$$

where $M = u/c_s$ is the Mach number of the flow, the ratio of flow velocity to the sound speed,

$$c_s^2 = \frac{dp}{d\rho}\,. \tag{10}$$

d. Invoking the continuity equation (question b), show that

$$\frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0 \tag{11}$$

e. and hence show that

$$(1 - M^2)\frac{du}{u} = -\frac{dA}{A}\,. \tag{12}$$

f. Investigating the consequences of this nozzle equation, describe first what the consequence is for the flow velocity as the cross section A changes and the flow is subsonic. On the other hand, what happens if the flow is supersonic. Why is the latter at first counterintuitive ? How can this be explained when looking at the development of the density ρ ?

g. A sonic transition happens when the flow passes from subsonic to supersonic, ie. when $M = 1$. If du/dx is finite, why does this happen at the throat of the nozzle ?

Question 3.: Gravity Waves

We consider gravity waves in a fluid of *finite* depth. In the hoorcollege of March 6 (week 4), we assumed a potential flow and that the vertical displacement ζ was very small. This gave rise to the equations

$$\begin{aligned} \Delta\Phi_v &= 0 \quad \text{Poisson equation} \qquad (13) \\ \left(\frac{\partial\Phi_v}{\partial z} + \frac{1}{g}\frac{\partial^2\Phi_v}{\partial t^2}\right)_{z=\zeta\approx 0} &= 0 \quad \text{boundary condition} \qquad (14) \end{aligned}$$

a. The depth of the fluid is h and take the fluid's surface at $z = 0$. Assume a potential flow. Which new boundary condition should we impose on the fluid (and the potential)?

b. We (still) expect a simple periodic function in time as our solution:

$$\Phi_v = f(z)\cos(kx - \omega t). \qquad (15)$$

Use Poisson's equation to find the general solution for $f(z)$ (which is slightly more general than in the infinite depth case, since z cannot be $-\infty$).

c. Using the boundary condition in question 2., show that

$$\Phi_v = A\cosh(kz + kh)\cos(kx - \omega t) \qquad (16)$$

for arbitrary A.

d. This velocity potential gives rise to a non-zero velocity-component in the x-direction at $z = -h$ (at the bottom of the fluid). What is the velocity in the x-direction?

e. Assume that we are talking about an ocean with sand on the bottom that we can (very) roughly describe as a fluid as well. We then have two fluids with different velocities and different densities, one moving over the other. How do we call such a situation in general? Make a rough sketch of the sand just after the gravity wave started (when it has only moved in one direction yet).

Question 4.: Numerical Hydrodynamics

In Problem Set 6 about numerical hydrodynamics schemes we tried to solve the advection equation:

$$\frac{\partial q}{\partial t} + u\frac{\partial q}{\partial x} = 0\,, \tag{17}$$

with a shock-like initial condition (a discontinuity). We found that in the first-order methods the shock was diffused, as illustrated in the accompanying figure where we see diffusion of the shock in the numerical Lax scheme at $t = 0.5$ with $u = 1$.

This unexpected behaviour is caused by the fact that while this method is a first order accurate approximation of the advection equation, it is also in fact a second order accurate approximation of a modified version of the original equation. In this modified equation we also have a diffusion term and so we are actually solving, to second order accuracy, the advection-diffusion equation for a fluid with homogeneous velocity u and diffusion coefficient D:

$$\frac{\partial q}{\partial t} + u\frac{\partial q}{\partial x} = D\frac{\partial^2 q}{\partial x^2}\,. \tag{18}$$

a. Give the general (non-homogeneous) form of the advection-diffusion equation.

b. The Lax method is a modified version of the "forward time, central space" (FTCS) scheme, where the q_i^n term is replaced by its (space) average:

$$q_i^n \to \frac{1}{2}\left(q_{i+1}^n + q_{i-1}^n\right)\,. \tag{19}$$

Write down the Lax-method approximation to the solution of the advection equation in the form $q_i^{n+1} = \ldots$. Remember that a central space differential is given by

$$\frac{\partial q_i}{\partial x} \approx \frac{q_{i+1}^n - q_{i-1}^n}{2\Delta x}\,, \tag{20}$$

and a forward time differential is given by

$$\frac{\partial q_i}{\partial t} \approx \frac{q_i^{n+1} - q_i^n}{\Delta t}\,. \tag{21}$$

In the Lax-method approximation equation, we can replace all numerical estimate terms q_i^n by the exact solution $q(x,t)$ ($q_{i+1}^n = q(x+\Delta x, t)$, etc.). The equation we are then left with is no longer equal to zero, but rather to

E_Δ which represents the error of the conversion from exact to discrete. Thus, we end up with an equation like E_Δ = "*numerical scheme with estimate terms replaced by exact terms*".

If we now Taylor expand all $q(x,t)$ terms around (x,t) up to second order we get for the error the following equation:

$$E_\Delta = \frac{\partial q}{\partial t} + u\frac{\partial q}{\partial x} + \frac{1}{2}\left(\frac{\partial^2 q}{\partial t^2}\Delta t - \frac{\partial^2 q}{\partial x^2}\frac{\Delta x^2}{\Delta t}\right) + O(\Delta x^3)\,. \tag{22}$$

c. Rewrite the equation for the error term E_Δ to something of the form:

$$E_\Delta = -D\frac{\partial^2 q}{\partial x^2} + O(\Delta x^3)\,. \tag{23}$$

Use $u = \frac{\partial x}{\partial t}$ to get rid of the second order time derivative.

Use this result to explain the diffusive behaviour of the Lax-method.

d. Give the Courant-Friedrichs-Lewy condition for the Lax numerical scheme (and other numerical schemes) to be stable.

e. For the Lax-method, the diffusion term is given by

$$D = \frac{\Delta x^2}{2\Delta t}\left(1 - \left(u\frac{\Delta t}{\Delta x}\right)^2\right)\,. \tag{24}$$

If we set $\Delta t = 0.5\Delta x$, why should we not worry about the diffusion term becoming negative (use the stability condition)?

SUCCES !!!!
BEDANKT VOOR JULLIE AANDACHT EN INTERESSE !!!!

Rien & Patrick